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Global Stability Results for the Weak Vector Variational Inequality

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Abstract. In this paper, we consider the global stability of solutions of a Weak Vector Variational Inequality in a finite-dimensional Euclidean space. Upper semi-continuity of the solution set mapping is established. And by a scalarization method, we derive a sufficient condition that guarantees the lower semi-continuity of the solution set mapping for the Weak Vector Variational Inequality.

Key words: Lower semi-continuity, Stability, Strictly pseudo-monotone function, Upper semi-continuity, Weak vector variational inequality

1. Introduction

A Vector Variational Inequality (VVI) in a finite-dimensional Euclidean space was firstly proposed by Giannessi (1980), which is the vector-valued version of the Variational Inequality (for short, VI) of Hartman and Stampachia (1966). Since then, a great deal of research has been devoted to the existence of solutions to VVI in various versions; see, e.g., Giannessi, Mastronei and Pellegrini (1999), Chen (1992), Chen and Craven (1990), Chen and Li (1996), Chen and Yang (1990), Lee, Kim, Lee and Yen (1998), Daniilidis and Nadjisavva (1996), Yang (1993) and references therein.

Among many desirable properties of the solution set for VVI, the stability analysis is of considerable interest. Nevertheless, until now, little work has been made in the study of the semi-continuity of the solution set for VVI, especially there exists no general lower semi-continuity result in the literature. Lower semi-continuity of the solution set of VVI is much stronger than upper semi-continuity. It is a much more difficult and challenging work to derive conditions that guarantee lower semi-continuity because of the complexity of the problem structure. This paper is tentatively dedicated to the global stability properties for a Weak Vector Variational Inequality ($WVVI$) with respect to perturbation of the underlying set and functions. Firstly, an upper semi-continuity result of the solution set mapping for the $WVVI$ is presented. More importantly, we use a scalarization method for analyzing the $WVVI$ and derive a sufficient condition that ensures lower semi-continuity of the solution set mapping of the $WVVI$.

The paper is organized as follows. In Section 2, we introduce the $WVVI$ and establish the upper semi-continuity of its solution set. The lower semi-continuity result of the solution set for the $WVVI$ is provided in Section 3.

2. Upper Semi-continuity

Let X be a nonempty, closed, and convex subset of $\mathbb{R}^n$, and let $f_i : X \to \mathbb{R}^n, i = 1, \dots, p$, be vector-valued functions. We define $f := (f_1, \dots, f_p)$ as follows: for every $x \in X$ and $y \in \mathbb{R}^n$,

$$f(x)(y) := (\langle f_1(x), y\rangle, \dots, \langle f_p(x), y\rangle),$$

where $\langle x, y\rangle$ denotes the inner product of the vectors x and y in the Euclidean space.

$$\mathbb{R}^p_+ = \{y \in \mathbb{R}^p : y_i \geqslant 0, \text{ for all } i = 1, \dots, p\}$$

and

$$\overset{\circ}{\mathbb{R}}{}^p_+ = \{y \in \mathbb{R}^p : y_i > 0, \text{ for all } i = 1, \dots, p\}$$

denote the non-negative and positive orthants of $\mathbb{R}^p$, respectively.

We denote $a \nless_{\overset{\circ}{\mathbb{R}}{}^p_+} 0$ for the vector $a \in R^p$ if there is, at least, $j \in \langle 1, p\rangle$ such that $a_p \geqslant 0$.

A $WVVI$ consists in finding $\bar{x} \in X$ such that

$$f(\bar{x})(x - \bar{x}) \nless_{\overset{\circ}{\mathbb{R}}{}^p_+} 0, \text{ for all } x \in X.$$

We denote by $S^w(f, X)$ the solution set of the $WVVI$.

When the set X and the functions $f_i, i = 1, \dots, p$, are perturbed by a parameter μ, which varies over a set Λ of $\mathbb{R}^\ell$ (the space of parameters), we can define the parameterized Weak Vector Variational Inequality $(WVVI)_\mu$: finding $\bar{x} \in X(\mu)$ such that

$$f(\bar{x}, \mu)(x - \bar{x}) \nless_{\overset{\circ}{\mathbb{R}}{}^p_+} 0, \text{ for all } x \in X(\mu).$$

Here, X is a set-valued mapping from Λ into $\mathbb{R}^n$, $f_i, i = 1, \dots, p$, are vector-valued functions on $\mathbb{R}^n \times \Lambda$, and

$$f(x, \mu)(y) = (\langle f_1(x, \mu), y\rangle, \dots, \langle f_p(x, \mu), y\rangle).$$

For each $\mu \in \Lambda$, let $S(\mu)$ denote the (possible) nonempty solution set of $(WVVI)_\mu$: $S(\mu) = \{\bar{x} \in X(\mu) | f(\bar{x}, \mu)(x - \bar{x}) \not\leq_{\mathring{\mathbb{R}}^p_+} 0, \ \text{for all } x \in X(\mu)\}$.

The problem of stability analysis in this paper deals with the investigation of the behavior of the solution set $S(\mu)$ as the parameter vector μ varies over the set Λ. We discuss the continuity (upper semi-continuity and lower semi-continuity) of $S(\cdot)$ as a set-valued mapping from the set Λ into R^n.

Let us first recall some notations and definitions that are used later on.

DEFINITION 2.1. A vector-valued function $g: \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone over a closed and convex set X if for any $x, y \in X$, $\langle g(y) - g(x), x - y \rangle \geqslant 0$.

DEFINITION 2.2. A vector-valued function $g: \mathbb{R}^n \to \mathbb{R}^n$ is said to be pseudo-monotone over a closed and convex set X if for any $x, y \in X$, $\langle g(y), x - y \rangle \geqslant 0$ implies $\langle g(x), x - y \rangle \geqslant 0$.

DEFINITION 2.3. A vector-valued function $g: \mathbb{R}^n \to \mathbb{R}^n$ is said to be strictly monotone over a closed and convex set X if for any $x, y \in X$, $x \neq y$, $\langle g(y) - g(x), x - y \rangle > 0$.

The reader interested in monotonicity and generalized monotonicity is referred to the paper of Crouzeix (1998) and Zhu and Marcotte (1995).

DEFINITION 2.4. Sawaragi et al. (1985). A set-valued mapping F from a set X to a set Y is said to be uniformly compact near a point $x \in X$, if there is a neighborhood U of x such that the closure of the set $\bigcup_{x \in U} F(x)$ is compact.

DEFINITION 2.5. Aubin and Ekeland (1984). A set-valued mapping G from a set X to a set Y is said to be:

(1) Upper semi-continuous at $x^* \in X$ if for any open set W containing $G(x^*)$, there exists a neighborhood U of x^* such that $G(x) \subset W$ for all $x \in U$.
(2) Lower semi-continuous at $x^* \in X$ if for any open set W intersecting $G(x^*)$, there exists a neighborhood U of x^* such that $G(x) \cap W \neq \emptyset$ for all $x \in U$.

We say that G is continuous at x^* if it is both lower and upper semi-continuous; and we say G is continuous in X if it is continuous at each point of X.

We now present the upper semi-continuity result of the solution set mapping $S(\cdot)$.

THEOREM 2.1. *If the following assumptions (a)–(d) hold, then* $S(\cdot)$ *is upper semi-continuous at* $\mu \in \Lambda$*:*

(a) $f_i, i = 1, \dots, p$, *are continuous;*
(b) $X: \rightrightarrows R^n$ *is continuous, convex-valued and compact valued;*
(c) $X(\cdot)$ *is uniformly compact near* $\mu \in \Lambda$*;*
(d) $f_i(\cdot, \mu), i = 1, \dots, p$, *are pseudomonotone.*

Proof. We first show that $S(\cdot)$ is a closed mapping.

Take $x_n \in S(\mu_n)$ with $\mu_n \to \mu$ and $x_n \to \bar{x}$. From the assumption (d), we have that

$$f(x, \mu_n)(x - x_n) \not\leq_{\mathring{\mathbb{R}}_+^p} 0, \text{ for all } x \in X(\mu_n). \tag{2.1}$$

Now, let us take any $y \in X(\mu)$. Construct one open set $W_0 = \{x \mid \|x - y\| < 1\}$. By the lower semi-continuity of the mapping $X(\cdot)$, there exists μ_{n_0} such that $X(\mu_{n_0}) \cap W_0 \neq \emptyset$. Take $y_0 \in X(\mu_{n_0}) \cap W_0$ and $y_0 \neq y$. Sequentially, we construct the open set $W_k = \left\{x \mid \|x - y\| < \frac{\|y_{k-1} - y\|}{2}\right\}, k = 1, 2, \dots$. Take μ_{n_k} such that $\|\mu_{n_k} - \mu\| < \frac{\|\mu_{n_{k-1}} - \mu\|}{2}$ such that $X(\mu_{n_k}) \cap W_k \neq \emptyset$, $y_k \in X(\mu_{n_k}) \cap W_k$, $y_k \neq y$. Thus from (2.1), we deduce that

$$f(y_k, \mu_{n_k})(y_k - x_{n_k}) \in W :\subset \mathbb{R}^p \setminus -\mathring{\mathbb{R}}_+^p. \tag{2.2}$$

Since $f = (f_1, \dots, f_p)$ is continuous, it follows that

$$f(y, \mu)(y - \bar{x}) \not\leq_{\mathring{\mathbb{R}}_+^p} 0, \text{ for all } y \in X(\mu) \tag{2.3}$$

which yields that $\bar{x} \in S(\mu)$.

Secondly, we prove that the mapping $S(\cdot)$ is upper semi-continuous. Otherwise, there would be an open set O with $S(\mu) \subset O$, for $\mu_n \to \mu$, there exist $y_n \in S(\mu_n)$, it holds that

$$y_n \notin O. \tag{2.4}$$

Then, from the definition of $S(\cdot)$, we have $y_n \in X(\mu_n)$. By the assumption that $X(\cdot)$ is uniformly compact near μ, there exists $y \in \mathbb{R}^n$ such that $y_n \to y$. Since $S(\cdot)$ is a closed mapping, we get $y \in S(\mu) \subset O$. Hence, there would be a natural number N such that $y_N \in O$. Thus, a contradiction appears.

This establishes upper semi-continuity of $S(\cdot)$. □

3. Lower Semi-continuity

In this section, we establish lower semi-continuity of the solution set mapping $S(\cdot)$ by a scalarization method. For this aim, we introduce a scalar version of the $WVVI$ and establish the equivalence between the $WVVI$ and a family of scalar VI.

Take $\xi \in S_+ := \{x \in \mathbb{R}^p_+ : \|x\| = 1\}$. In view of the $WVVI$, we consider the following scalar Variational Inequality $(VI)_\xi$: finding $\bar{x} \in X$ such that

$$\langle F(\bar{x}, \xi), x - \bar{x} \rangle \geqslant 0, \text{ for all } x \in X,$$

where $F(\bar{x}, \xi) = \sum_{i=1}^p \xi_i f_i(\bar{x})$. And the parameterized Variational Inequality $(VI)_{(\xi,\mu)}$ with respect to $(WVVI)_\mu$ consists in finding $\bar{x} \in X(\mu)$ such that

$$\langle F(\bar{x}, \mu, \xi), x - \bar{x} \rangle \geqslant 0, \text{ for all } x \in X(\mu),$$

where $F(\bar{x}, \mu, \xi) = \sum_{i=1}^p \xi_i f_i(\bar{x}, \mu)$.

We denote by $I_\xi(\mu)$ the (possible) nonempty solution set of $(VI)_{(\xi,\mu)}$. Then we have the following lemma.

LEMMA 3.1. *If $X(\cdot)$ is convex-valued, then for all $\mu \in \Lambda$,*

$$S(\mu) = \bigcup_{\xi \in S_+} I_\xi(\mu).$$

Proof. $\bar{x} \in S(\mu)$ yields that

$$\{f(\bar{x}, \mu)(x - \bar{x}) : x \in X(\mu)\} \cap -\overset{\circ}{\mathbb{R}}{}^p_+ = \emptyset. \tag{3.1}$$

Since $X(\cdot)$ is convex-valued, it is clear that the set $\{x | f(\bar{x}, \mu)(x - \bar{x}) : x \in X(\mu)\}$ is convex. Then by the separation theorem for convex sets there exists $\xi \in R^p \backslash \{0\}$ such that

$$\inf_{x \in X(\mu)} \langle \xi, f(\bar{x}, \mu)(x - \bar{x}) \rangle \geqslant \sup\nolimits_{y \in -\overset{\circ}{\mathbb{R}}{}^p_+} \langle \xi, y \rangle. \tag{3.2}$$

This implies that $\xi \in R^p_+ \backslash \{0\}$ and $\langle \xi, f(\bar{x}, \mu)(x - \bar{x}) \rangle \geqslant 0$. Let $\bar{\xi} = \xi / \|\xi\|$. Then, $\bar{\xi} \in S_+$ and $\langle F(\bar{x}, \mu, \bar{\xi}), x - \bar{x} \rangle \geqslant 0$ for all $x \in X(\mu)$. This amounts to saying that $\bar{x} \in I_{\bar{\xi}}(\mu)$.

Conversely, assume that $\bar{x} \in I_\xi(\mu)$ where $\xi \in S_+$. Then

$$0 \leqslant \left\langle \sum_{i=1}^{p} \xi_i f_i(\bar{x}, \mu), x - \bar{x} \right\rangle = \sum_{i=1}^{p} \xi_i \langle f_i(\bar{x}, \mu), x - \bar{x} \rangle = \langle \xi, f(\bar{x}, \mu)(x - \bar{x}) \rangle$$

implies that $f(\bar{x}, \mu)(x - \bar{x}) \not\leqslant_{\overset{\circ}{\mathbb{R}}{}^p_+} 0$.

This establishes the lemma. □

LEMMA 3.2. *Assume that*

(a) $f_i, i = 1, \ldots, p$, *are continuous;*
(b) $X : \rightrightarrows R^n$ *is continuous, convex-valued and compact valued.*
(c) $X(\cdot)$ *is uniformly compact near* $\mu \in \Lambda$;
(d) $f_i(\cdot, \mu), i = 1, \ldots, p$, *are monotone and there exists one strictly monotone mapping* $f_j(., \mu)$, $j \in \langle 1, p \rangle$.

Then $I_\xi(\mu)$ *is nonempty, singleton and continuous at* μ.

Proof. (I). The assertion that $I_\xi(\mu)$ is nonempty holds clearly.
(II). We now prove that $I_\xi(\mu)$ is a singleton.
If not, take $y, z \in I_\xi(\mu)$, $y \neq z$. We have $y, z \in X(\mu)$ and

$$\langle F(y, \mu, \xi), z - y \rangle \geqslant 0. \tag{3.3}$$

and

$$\langle F(z, \mu, \xi), y - z \rangle \geqslant 0. \tag{3.4}$$

Therefore, we obtain

$$\langle F(y, \mu, \xi) - F(z, \mu, \xi), y - z \rangle \leqslant 0.$$

From the assumption (d), the mapping $F(x, \mu, \xi)$ is strictly monotone and we have

$$\langle F(y, \mu, \xi) - F(z, \mu, \xi), y - z \rangle > 0.$$

which follows the contradiction.

(III). Finally, we prove that $I_\xi(\cdot)$ is continuous at μ.

For this, assume that $\mu_n \to \mu$ and $I_\xi(\mu) = \bar{x}$. Then by the definition of $I_\xi(\mu)$, it follows that $\bar{x} \in X(\mu)$ and

$$\langle F(\bar{x}, \mu, \xi), x - \bar{x} \rangle \geqslant 0, \text{ for all } x \in X(\mu). \tag{3.5}$$

Since X is lower semi-continuous, there exist $x_n \in X(\mu_n)$ with $x_n \to \bar{x}$.

Since X is convex-valued, compact-valued and f_i are continuous, we know $I_\xi(\mu_n) \neq \emptyset$. Let $z_n \in I_\xi(\mu_n)$. Then $z_n \in X(\mu_n)$ and

$$\langle F(z_n, \mu_n, \xi), x - z_n \rangle \geqslant 0, \text{ for all } x \in X(\mu_n).$$

From $x_n \in X(\mu_n)$, we have

$$\langle F(z_n, \mu_n, \xi), x_n - z_n \rangle \geqslant 0 \tag{3.6}$$

And since X is uniformly compact near μ, $z_n \to z$. Hence, we obtain, from the continuity of X, that $z \in X(\mu)$. In View of $\bar{x} = I_\xi(\mu)$, we have

$$\langle F(\bar{x}, \mu, \xi), z - \bar{x} \rangle \geqslant 0. \tag{3.7}$$

By the continuity of f_i and (3.6), we obtain that

$$\langle F(z, \mu, \xi), \bar{x} - z \rangle \geqslant 0. \tag{3.8}$$

Combining (3.7) and (3.8), from the monotonicity of $F(\cdot, \mu, \xi)$ we have

$$\langle F(\bar{x}, \mu, \xi) - F(z, \mu, \xi), \bar{x} - z \rangle \leqslant 0.$$

This yields that $z = \bar{x}$ from the strictly monotonicity of $F(\cdot, \mu, \xi)$.

Thus we have showed that there exist $z_n = I_\xi(\mu_n)$ with $z_n \to \bar{x}$. This establishes continuity of $I_\xi(\cdot)$ at $\mu \in \Lambda$.

The proof is completed. □

Now, we will present the lower semi-continuity result of the solution set mapping $S(\cdot)$.

THEOREM 3.1. If the assumptions (a)–(d) in Lemma 3.2 hold, then $S(\cdot)$ is lower semi-continuous at $\mu \in \Lambda$.

Proof. Firstly, $S(\mu)$ is nonempty.

Take $\mu_n \to \mu$ and $x \in S(\mu) = \bigcup_{\xi \in S_+} I_\xi(\mu)$. Then there exists $\xi \in S_+$ such that $x = I_\xi(\mu)$. By Lemma 3.2, we know that I_ξ is continuous at μ. Hence, there exist $x_n = I_\xi(\mu_n)$ such that $x_n \to x$. Note that $I_\xi(\mu_n) \subset S(\mu_n)$. Then we complete the proof. □

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